Section 7.5 part 1

7.5 The Symmetric and Alternating Groups

Cor 7.22. Every finite group is isomorphic to a subgroup of S_n

Recall S_n - all bijective functions (permutations)

$$\{1,2,\dots,n\} \to \{1,2,\dots,n\}$$

group operation is the composition of functions

Notation for $\sigma \in S_n$: $\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$

Cycles - special elements of S_n

Notation: $\sigma = (a_1\, a_2 \dots a_k)$ $\quad a_1 \dots a_k$ are all distinct elements of the set $\{1, \dots n\}$

$k \leq n$

Meaning: $\sigma(a_1) = a_2 \quad \sigma(a_2) = a_3 \dots \sigma(a_{k-1}) = a_k, \ \sigma(a_k) = a_1$

$$a_1 \mapsto a_2 \mapsto a_3 \mapsto \dots \mapsto a_k$$

All other elements stay in their places:

$\sigma(b) = b$ if $b \neq a_i$

Easy to see: $|(a_1, \dots a_k)| = k$

- order of this element of S_n

Def $\sigma = (a_1, \ldots, a_k)$ and $\tau = (b_1, \ldots, b_r)$
are *disjoint* when they do not share elements.

Ex (125) and (34) are two disjoint cycles in S_5 (also S_{17})

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix}$$

Th 7.24 Every permutation can be written as a product of disjoint cycles in a unique way.

Th 7.23 Disjoint cycles commute:
if $\sigma, \tau \in S_n$ are disjoint cycles, then $\sigma\tau = \tau\sigma$.

Pf $\sigma = (a_1, \ldots, a_k)$ $\tau = (b_1, \ldots, b_r)$

Let $1 \le x \le n$. Wanted: $\sigma(\tau(x)) = \tau(\sigma(x))$

If x is neither a_i nor b_j, then $\sigma(x) = x$; $\tau(x) = x$
$\sigma(\tau(x)) = \tau(\sigma(x)) = x$

If $x = a_i$, then $x \neq b_j$. $\sigma(x) = a_l$ ($l = i+1$ or $l = 1$) $\tau(x) = x$
$\tau(a_i) = a_i$

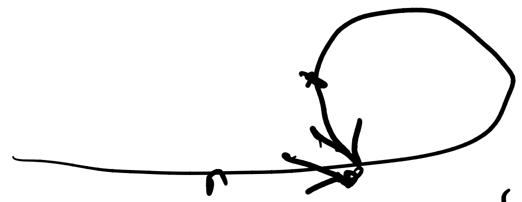
$\sigma(\tau(x)) = \sigma(x) = a_\ell$

$\tau(\sigma(x)) = \tau(a_\ell) = a_\ell$

Pf (7.24) $\sigma \in S_n$

$a_1 \in \{1, \dots, n\}$ $\quad a_2 = \sigma(a_1) \quad a_3 = \sigma(a_2) \quad \dots$

$$a_1 \overset{\sigma}{\longmapsto} a_2 \overset{\sigma}{\longmapsto} a_3 \dots a_k \overset{\sigma}{\longmapsto} a_1$$



would contradict the injectivity of σ

Pick another element b_1, not from this cycle

$$b_1 \overset{\sigma}{\longmapsto} b_2 \overset{\sigma}{\longmapsto} \dots \overset{\sigma}{\longmapsto} b_r \overset{\sigma}{\longmapsto} b_1$$

...

When all elements from $\{1, \dots, n\}$ are either members of a cycle or are left on their places by σ, we see that σ is the product of the cycles which we constructed.

Transpositions - cycles of length 2

(a, b)

$$\begin{pmatrix} \ldots & a & \ldots & b & \ldots \\ \ldots & b & \ldots & a & \ldots \end{pmatrix}$$

$a \mapsto b$

$b \mapsto a$

Th 7.26 Every permutation can be presented as a product of transpositions. (not necessarily disjoint)

Pf It suffices to present a cycle as a product of transpositions:

$$(a_1, \ldots a_k) = (a_1 a_2)(a_2 a_3) \ldots (a_{k-1} a_k)$$

$a_1 \mapsto a_2$ $\quad$ $a_1 \mapsto a_2$

$a_1 \mapsto a_3$ $\quad$ $a_2 \mapsto a_3$

$a_k \mapsto a_1$

Def A permutation is EVEN if it can be presented as a product of an even number of transpositions

Otherwise - ODD permutation.

Th 7.28 This parity is invariant (the definition is justified)